# ON THE THEORY OF FUGOID MOTIONS

## (K TEORII FUGOIDNYKH DVIZHENII)

PMM Vol.30, № 6, 1966, pp.1106-1111

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(Received February 16, 1966)

In 1891 Zhukovslii in his paper "On soaring of birds" [1] solved the problem of the motion of a body of high lift – drag ratio in an atmosphere of constant density. In [2] this problem was considered in greater detail, but the basic assumption of a constant density was made here as well. There have recently appeared numerous papers concerning the analytical solution of the problem of entry into the atmosphere with orbital and escape velocities [3 to 5]. But these studies were concerned primarily with the problems of ballistic entry and entry with low lift – drag ratio. In considering oscillatory states, the authors limited their treatment to small angles between the trajectory and local horizon. In the present paper we consider the problem without imposing any limitations on the slope of the trajectory or initial velocity. The case examined will be that of a hypothetical glider spacecraft of sufficiently high lift – drag ratio. It is interesting to note that the solution of this problem reduces to the solution of Zhukovskii's problem, but for an atmosphere of variable density. The associated trajectories are termed "fugoid". All of our assumptions about the parameters of such a glider are of a particular hypothetical character.

**1, Statement of the problem.** Let us consider the motion of a body of high lift – drag ratio in a great circle plane of a spherical nonrotating planet with an isothermal atmosphere.

The motion of such a body is described by the following system of equations:

$$\frac{dv}{dt} = -g\sin\theta - \frac{c_x S}{2m}\rho v^2, \qquad \frac{dH}{dt} = v\sin\theta$$

$$\frac{d\theta}{dt} = \cos\theta\left(\frac{v}{R+H} - \frac{g}{v}\right) + \frac{kc_x S}{2m}\rho v, \qquad \frac{dL}{dt} = v\cos\theta \tag{1.1}$$

$$\left(\rho = \rho_0 e^{-\lambda H}, \quad g = g_0\frac{R^2}{(R+H)^2}\right)$$

with the initial conditions

$$t = 0, \quad v = v^\circ, \quad \theta = \theta^\circ, \quad H = H^\circ, \quad L = 0$$

Here $\rho_0$, $g_0$ are the atmospheric density and gravitational acceleration at the planet's surface, respectively; $\lambda$ is the atmosphere index, $v$ is the absolute value of the velocity vector in the associated coordinate system (Fig.1); $\theta$ is the angle between the trajectory and local horizon; $H$ is the altitude of the body above the planet's surface; $L$ is the distance flown as measured along the great circle arc; $R$ is the radius of the planet; $m$, $S$, $C_x$, $k$ are the mass of the body, the body reference area, the drag coefficient and the lift – drag ratio of the body, respectively. These quantities are assumed constant.

We shall consider the portion of the trajectory having the following property: the altitude of the body is much less than the radius of the planet $(H \ll R)$ . Then with sufficent accouracy we can assume that

$$R + H \approx R, \qquad g \approx g_0 \tag{1.2}$$

Let us introduce the following dimensionless variables:

$$n = \frac{C_x S}{2mg_0}\rho v^2, \quad V = \frac{v}{\sqrt{Rg_0}}, \quad \tau = \left(\frac{g_0}{R}\right)^{1/2} t, \quad h = \frac{H}{R}, \quad l = \frac{L}{R}, \quad \beta = \lambda R \tag{1.3}$$

Substituting (1.3) into (1.1) and taking account of (1.2), we find that

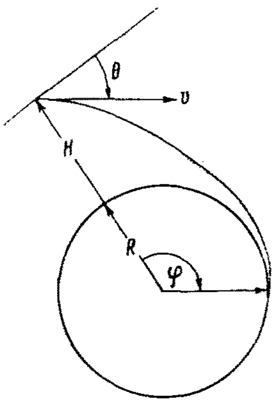

Fig. 1

$$\frac{dn}{d\tau} = -n\sin\theta\left(\beta V + \frac{2}{V}\right) - \frac{2n^2}{V}$$

$$\frac{dV}{d\tau} = -\sin\theta - n$$

$$V\frac{d\theta}{d\tau} = \cos\theta\,(V^2 - 1) + kn \tag{1.4}$$

$$\frac{dh}{d\tau} = V\sin\theta, \quad \frac{dl}{d\tau} = V\cos\theta$$

with initial conditions

$$\tau = 0, \quad n = n^\circ, \quad \theta = \theta^\circ, \quad V = V^\circ, \quad h = h^\circ, \quad l = 0$$

The first three equations of (1.4) form a closed system. The solutions for $h$ and $l$ if $n$, $V$, $\theta$ are known are obtained in quadratures.

We shall consider a body of high lift – drag ratio and low drag. Physically, this means that the system is nearly conservative and that the dissipative forces are small. On the other hand, the lift, which curves the trajectory without doing work, plays a substantial role. A system of this type was considered in [1, 2 and 5].

Upon entry of the spacecraft into the planet's atmosphere, the axial overload (and hence the drag) must be small, since the thermal shielding and crew tolerance are restrictive factors. The maximum overload can be reduced by effecting entry with a lift – drag ratio which can attain quite substantial values [3]. It is obvious, therefore, that the present problem is of immediate practical interest.

**2. Investigation of the zeroth approximation.** We introduce the small parameter $\epsilon > 0$ as follows. Let the quantities

$$n = \varepsilon N, \qquad K = \varepsilon k \tag{2.1}$$

be of the order of unity. In realty, $\epsilon$ is equal in order of magnitude to the ratio of the energy released over the flight time in the form of heat, to the total energy of the body at the instant of entry. Usually this ratio is of the order of $10^{-1}$ or smaller. On the other hand, $\epsilon^2$ can be defined as the ratio of the mean overload during flight time to the lift – drag ratio.

Substituting (2.1) into (1.4) and making $\epsilon$ go to zero, we obtain the generating system for (1.4) in the form

$$\frac{dN_0}{d\tau} = -N_0\sin\theta_0\left(\beta V_0 + \frac{2}{V_0}\right), \qquad V_0\frac{d\theta_0}{d\tau} = \cos\theta_0\,(V_0^2 - 1) + KN_0$$

$$\frac{dV_0}{d\tau} = -\sin\theta_0, \qquad \frac{dl_0}{d\tau} = V_0\cos\theta_0, \qquad \frac{dh_0}{d\tau} = V_0\sin\theta_0 \tag{2.2}$$

under the initial conditions

$$\tau = 0, \quad N_0 = N^\circ, \quad \theta_0 = \theta^\circ, \quad V_0 = V^\circ, \quad h_0 = h^\circ, \quad l_0 = 0$$

Here $N_0$, $V_0$, $\theta_0$, $l_0$, $h_0$ is the solution of the generating system. Let us consider the first three equations of (2.2), which form a closed system. This system has integrals of the form

$$\frac{N_0}{V_0^2}\exp\left(-\frac{\beta V_0^2}{2}\right)=c_1=\text{const}$$

$$\cos\theta_0 V_0\exp\left(-\frac{V_0^2}{2}\right)-Kc_1\int_{V^\circ}^{V_0}V_0^2\exp\left(\frac{\beta-1}{2}V_0^2\right)dV_0=c_2=\text{const}$$

$$\int_{V^\circ}^{V_0}\frac{V_0\,dV_0}{\sqrt{V_0^2-(c_2+Kc_1J)^2\exp V_0^2}}+\tau=c_3=\text{const}\qquad(2.3)$$

Here

$$J=\int_{V^\circ}^{V_0}V_0^2\exp\left(\frac{\beta-1}{2}V_0^2\right)dV_0\qquad(\beta\gg 1)$$

Let us analyze the qualitative picture of the motion determined by the solutions of the generating system. We consider the first two integrals from (2.3). Relating $c_1$ and $c_2$ to the initial conditions, we obtain

$$\cos\theta_0=\frac{e^{\alpha_1(\xi^2-1)}}{\xi}\left\{kn^0\int_1^{\xi}\xi^2e^{\alpha_2(\xi^2-1)}\,d\xi+\cos\theta^0\right\}\qquad(2.4)$$

$$\xi=V_0\,/\,V^\circ,\qquad \alpha_1={}^1/_2(V^\circ)^2\qquad \alpha_2={}^1/_2\,(\beta-1)\,(V^\circ)^2$$

Since $(V^\circ)^2\sim 1$, and since $\beta\sim 10^3$ for the planets of the terrestrial group (Venus, Earth, Mars), we estimate the integral in (2.4) by expanding it asymptotically in the parameter $1/\alpha_2=2\,/\,((\beta-1)\,V^\circ)^2\sim{}^1/_{500}$.

Integrating by parts, we obtain

$$e^{-\alpha_2}\int_1^{\xi}\xi^2e^{\alpha_2\xi^2}d\xi=\frac{1}{2\alpha_2}\left\{\xi e^{\alpha_2(\xi^2-1)}-1\right\}+O\,(\alpha_2^{-2})\qquad(2.5)$$

Substituting (2.5) into (2.4) and limiting ourselves to terms of the order $O(\alpha_2^{-1})$, we have

$$\cos\theta_0\approx\frac{e^{\alpha_1(\xi^2-1)}}{\xi}\left\{\frac{kn^\circ}{2\alpha_2}\,[\xi e^{\alpha_2(\xi^2-1)}-1]+\cos\theta^\circ\right\}\qquad(2.6)$$

Let us consider the behavior of the two-parameter curves $\cos\theta_0=f(\xi,\cos\theta^\circ,kn^\circ)$ on the phase plane $\theta_0$, $\xi$ as a function of the values of the parameters and of $\cos\theta^\circ$, $kn^\circ$.

The parameter $kn^\circ$ determines the initial lift value. Fig.2 shows the

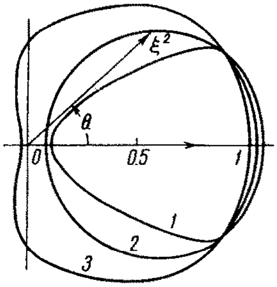


Fig. 2

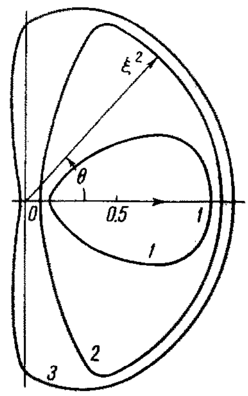


Fig. 3

family of curves $\xi = f(\theta_0)$ for $\cos\theta^\circ = 0.9$ ; moreover, $V^\circ = \sqrt{2}$ , which corresponds to entry with the escape velocity. Curves 1, 2, 3 correspond to the values $kn^\circ = 2,\ \alpha_2,\ 2\alpha_2$ .

Fig.3 shows the family of curves $\xi = f(\theta_0)$ for $kn^\circ = 2$, $V^\circ = \sqrt{2}$ for the three values $\cos\theta^\circ = 1,\ \frac{1}{2},\ 0$ .

The phase diagrams show that the motion is periodic with respect to velocity. (The phase trajectories are closed).

The curves in Figs.2 and 3 are shown in polar coordinates.

We see from these curves that the trajectories can be broken down into two types.

1) In the case $-\frac{1}{2}\pi < \theta_0 < \frac{1}{2}\pi$ the motion is purely oscillatory. Such motions will be called "fugoid". Fig.4 a shows the trajectory of such a flight. The corresponding phase trajectories are 1, 2 in Figs.2 and 3. It is clear that trajectories of this type occur upon fulfilment of the condition

$$\cos\theta_0 > 0 \qquad (2.7)$$

It then follows from (2.6) that

$$\frac{e^{\alpha_1(\xi^2-1)}}{\xi}\left\{\frac{kn^\circ}{2\alpha_2}\left[\xi e^{\alpha_2(\xi^2-1)} - 1\right] + \cos\theta^\circ\right\} > 0 \qquad (2.8)$$

Since $e^{\alpha_1(\xi^2-1)} / \xi > 0$ for any $\xi > 0$ , then (2.8) is equivalent to the condition

$$\frac{kn^\circ}{2\alpha_2}\xi e^{\alpha_2(\xi^2-1)} + \cos\theta^\circ > \frac{kn^\circ}{2\alpha_2} \qquad (2.9)$$

On entry into the atmosphere $\cos\theta^\circ$ is always positive, so that for $\xi > 1$ condition (2.9) is always fulfilled. For $\xi < 1$ the term

$$\frac{kn^\circ}{2\alpha_2}\xi e^{\alpha_2(\xi^2-1)}$$

is small as compared with unity. Hence, in this case condition (2.9) is equivalent to the condition

$$\cos\theta^\circ > \frac{kn^\circ}{2\alpha_2} \quad \text{or} \quad \cos\theta^\circ > \frac{kn^\circ}{(\beta-1)v^{\circ 2}} R g_0,$$

$$\alpha_2 = \frac{(\beta-1)v^{\circ 2}}{2Rg_0} \qquad (2.10)$$

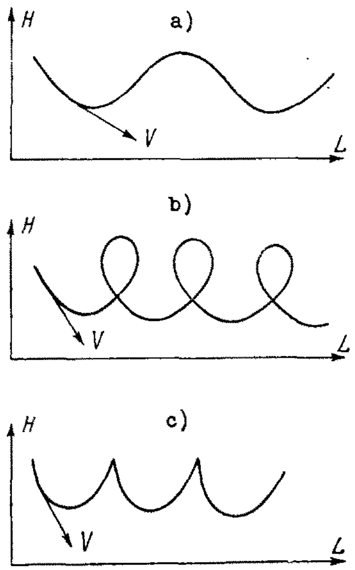


Fig. 4

2) On certain portions of the trajectory $|\theta_0| \geq \frac{1}{2}\pi$. In this case the velocity vector rotates. The approximate shape of such trajectories is shown in Figs. 4b and 4c ; these will be termed "loop" trajectories. It is clear that such trajectories occur upon fulfilment of the condition that

$$\frac{kn^\circ}{2\alpha_2}\xi e^{\alpha_2(\xi^2-1)} + \cos\theta^\circ \leqslant \frac{kn^\circ}{2\alpha_2} \qquad (2.11)$$

for some $\xi$ $\cos\theta_0 \leq 0$ (i.e. for some $\xi$ ) .

It is easy to see that upon fulfilment of the condition

$$\cos\theta^\circ \leqslant \frac{kn^\circ}{2\alpha_2} \qquad (2.12)$$

there will always be a $\xi$ ($0 < \xi < 1$) such that condition (2.11) is fulfilled. Just as in the above case, (2.12) can be written as

$$\cos\theta^\circ \leqslant \frac{kn^\circ}{(\beta-1)v^{\circ 2}} R g_0 \qquad (2.13)$$

Finally, the following statement can be formulated: occurrence of a "loop" trajectory requires fulfilment of condition (2.13); upon fulfilment of condition (2.10), the trajectory will be of the fugoid type.

We note that the system originally considered is dissipative, so that after a time the "loop" motion becomes fugoid, the latter in turn becoming motion wherein the trajectory inclination angle varies monotonously. Hence, the foregoing conditions are merely necessary.

In conclusion, let us write out the expressions for the maximum and minimum velocities for these types of trajectories.

It is easy to see that the maximum and minimum velocities can be determined from Equation

$$|\cos\theta_0| = 1 \qquad (2.14)$$

From (2.6) we find that

$$\left|\frac{kn^\circ}{2\alpha_2}\left[\xi^* e^{\alpha_2(\xi^{*2}-1)} - 1\right] + \cos\theta^\circ\right| = \xi^* e^{-\alpha_1(\xi^{*2}-1)} \qquad (2.15)$$

Here $V^\circ\xi^*$ is the extremal velocity. For "loop" trajectories (2.15) breaks down into two equations:

that for finding the maximum velocity

$$\frac{kn^\circ}{2\alpha_2}\left[\xi^* e^{\alpha_2(\xi^{*2}-1)} - 1\right] + \cos\theta^\circ = \xi^* e^{\alpha_1(1-\xi^{*2})} \qquad (\xi^* > 1) \qquad (2.16)$$

and that for finding the minimum velocity

$$\frac{kn^\circ}{2\alpha_2}\left[1 - \xi^* e^{\alpha_2(\xi^{*2}-1)}\right] + \cos\theta^\circ = \xi^* e^{\alpha_1(1-\xi^{*2})} \qquad (\xi^* < 1) \qquad (2.17)$$

If the trajectory under consideration corresponds to fugoid motion, we have a single equation which must have not less than two solutions corresponding to the maximum and minimum velocities,

$$\frac{kn^\circ}{2\alpha_2}\left[\xi^* e^{\alpha_2(\xi^{*2}-1)} - 1\right] + \cos\theta^\circ = \xi^* e^{\alpha_1(1-\xi^{*2})} \qquad (2.18)$$

In order to find the approximate solution of (2.16) we assume that $\xi^*$ is not much different from unity. (The phase trajectories shown in Figs. 2 and 3 show that such an assumption is quite realistic). The approximate value of $\xi^*$ is then given by Formula

$$\xi^* \approx 1 + \frac{2\alpha_2(1-\cos\theta^\circ)}{kn^\circ + 2\alpha_2(kn^\circ - 1) + 4\alpha_1\alpha_2}$$

which can be written as

$$v_{\max} \approx v^\circ\left\{1 + \frac{(\beta-1)(v^{\circ 2}/Rg_0)(1-\cos\theta^\circ)}{kn^\circ + (\beta-1)(v^{\circ 2}/Rg_0)(kn^\circ-1) + (\beta-1)(v^{\circ 2}/R^2g_0^2)}\right\} \qquad (2.19)$$

In attempting to find the approximate solution of (2.18) we assume that $\xi^*$ is small.

Then, neglecting the term $\xi^*\exp[\alpha_2(\xi^{*2}-1)]$ as compared with unity and neglecting terms of the order of $\xi^{*2}$, we obtain

$$\xi^* \approx e^{-\alpha_1}\left(\frac{kn^\circ}{2\alpha_2} + \cos\theta^\circ\right), \quad \text{or} \quad v_{\min} \approx v^\circ \exp\left(\frac{-v^{\circ 2}}{2Rg_0}\right)\left[\frac{kn^\circ Rg_0}{(\beta-1)v^{\circ 2}} + \cos\theta^\circ\right] \qquad (2.20)$$

In solving Equation (2.18) approximately, we assume that $\xi_{1,2}^*$ can be represented in the form $\xi_{1,2}^* = 1 \mp \delta$, where $\delta$ is small. Then, neglecting terms of the order of $\delta^2$ and higher, we obtain

$$\xi_{1,2}^* \approx 1 \mp \frac{2\alpha_2(1-\cos\theta^\circ)}{kn^\circ + 2\alpha_2(kn^\circ-1) + 4\alpha_1\alpha_2} \qquad (2.21)$$

or

$$v^*_{1,2} \approx v^\circ\left\{1 \mp \frac{(\beta-1)(v^{\circ 2}/Rg_0)(1-\cos\theta^\circ)}{kn^\circ + (\beta-1)(v^{\circ 2}/Rg_0)(kn^\circ-1) + (\beta-1)(v^{\circ 4}/R^2g_0^2)}\right\} \qquad (2.22)$$

Here the plus sign corresponds to the maximum, and the minus sign to the minimum velocity.

Knowing the first integral of the motion, we can use the averaging technique presented in [6] to construct the next approximation, which makes it possible to evaluate the effect of dissipative terms on the trajectory. Because of the analytical difficulties involved in constructing the next approximation, the latter will not be examined in the present paper. For this reason, the problem as to the character of the transition of the trajectory from the "loop" type to an ordinary fugoid one and the subsequent damping of the fugoid oscillations remains unsolved. Not having an analytic expression for time of motion along each of the "loop" motion, we are also unable to specify the number of such loops which the spacecraft executes before the motion becomes one of the ordinary fugoid type.

We note that the derived conditions for the realization of each given type of motion are merely necessary, since the effect of dissipative forces may be large enough to produce a transition from "loop" motion to fugoid motion in the very first loop, and a transition from the latter to motion with monotonous variation of the trajectory angle during the very first oscillation. The sufficient conditions can be obtained by analyzing the equations of the first approximation.

On the other hand, it is very interesting that the entry of a spacecraft into the atmosphere can involve "loop" motions in addition to the conventional fugoid oscillations which have been investigated in detail. Since the qualitative nature of these motions is the same as that of ordinary fugoid ones, we include them in the broader class of general fugoid motions.

The author is grateful to N.N. Moiseev for valuable advice and discussions.

## BIBLIOGRAPHY

1. Zhukovskii, N.E., O parenii ptits (On the soaring of birds). Collected works, Vol.4, Gostekhizdat, Moscow-Leningrad, 1949.

2. Ikonnikov, E., On the dynamics of symmetrical flight of an aeroplane. Tech.Phys. of the USSR, Vol.4, № 6, 1937.

3. Loh, T.H.E., Some analytical solutions of the re-entry problem. Repts. of the XIIIth International Congress of Astronautics. Varna, 1962.

4. Evtushenko, Iu.G., Asimptoticheskie metody integrirovaniia urvnenii dvizheniia iskusstvennykh sputnikov Zemli pri nalichii aerodinamicheskikh vozdeistvii (Asymptotic methods for integrating the equations of motion of artificial Earth satellites in the presence of aerodynamic forces). *PMM* Vol.29, № 3, 1965.

5. Iaroshevskii, V.A., Primenenie asimptoticheskogo metoda k nekotorym zadacham dinamiki letatel'nykh apparatov (Application of the asymptotic method to certain problems of aircraft dynamics). Inzh.Zh., Vol.2, № 2, 1962.

6. Volosov, V.M., Nekotorye vidy raschetov v teorii nelineinykh kolebanii, sviazannye s usredneniem (Certain types of calculations in the theory of nonlinear oscillations which involve averaging). Zh.vychisl.mat. Fiz., Vol.3, № 1, 1963.

Translated by A.Y.